UNIT - 1 - Discrete Fourier Transform

DFT and its properties, Relation between DTFT & DFT, FFT computations using DIT, DIF algorithm, Overlap - add and save methods.

DFT and its properties

DFT - Discrete fourier Transform

DFT computes the values of Z transform for evenly spaced points around the unit circle for a given sequence.

If the sequence to be represented is of finite duration ie has only a finite number of non-zero values, the transform used is DFT.

Applications of DFT

1. Linear filtering
2. Correlation Analysis
3. Spectrum Analysis

Definition for DFT (or) DFT pair

Let $x(n)$ be a finite duration sequence ~~Bes~~ N-point DFT of the sequence $x(n)$ is expressed by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}, \quad k=0,1 \ldots N-1$$

The IDFT is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi nk}{N}}, \quad n=0,1 \ldots N-1$$

Properties of DFT

1. **Periodicity :**

If $X(k)$ is an N-Point DFT of $x(n)$ then

$x(n+N) = x(n)$ for all n

$X(k+N) = X(k)$ for all k.

2. **Linearity :**

If $X_1(k)$ and $X_2(k)$ are the the N-point DFTs of $x_1(n)$ & $x_2(n)$ and a, b are arbitrary constants either real or complex valued then

$$a x_1(n) + b x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} a X_1(k) + b X_2(k)$$

3. **Shifting** - Circular Shifting.

Let $x_p(n)$ is a periodic sequence with period which is obtained by extending $x(n)$ periodically

ie $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$

if $DFT[x(n)] = X(k)$

$$DFT[x((n-l))_N] = e^{-j2\pi kl/N} X(k)$$

4. **Time Reversal of a sequence**

The time reversal of a N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in clock wise direction. It is denoted as $x((-n))_N$ and

5. Circular Convolution

If $x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$

$x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$

then $x_1(n) \; (N) \; x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k) X_2(k)$

circular convolution.

6) Circular frequency shift

If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$ then

$x(n) e^{-j 2\pi l n / N} \underset{N}{\overset{DFT}{\longleftrightarrow}} X((k-l))_N$

7) Complex conjugate Properties:

If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$ then

$x^*(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k)$

8) Circular Correlation.

For complex valued sequences $x(n)$ and $y(n)$ if

~~DFT [x(n)] = X(k)~~ and

~~DF~~ $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$

$y(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} Y(k)$

then $\tilde{r}_{xy}(l) \underset{N}{\overset{DFT}{\longleftrightarrow}} \tilde{R}_{xy}(k) = X(k) \, Y^*(k)$

Where $\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n) \, y^*((n-l))_N$

10) Parseval's Theorem:

For complex valued sequences $x(n), y(n)$

in general if

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

and $$y(n) \xleftrightarrow[N]{DFT} Y(k)$$

then $$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

11. Symmetry properties — (To be inserted after Linearity property)

N-point sequence $x(n)$ $0 \le n \le N-1$	N-point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$
$j x_I(n)$	$X_{co}(k) = \frac{1}{2}\left[X(k) - X^*(N-k)\right]$
$x_{ce}(n) = \frac{1}{2}\left[x(n) + x^*(N-n)\right]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}\left[x(n) - x^*(N-n)\right]$	$j X_I(k)$
Real Signals	
Any real signal	$X(k) = X^*(N-k)$ $X_R(k) = X_R(N-k)$ $X_I(k) = -X_I(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$

Comparison b/w linear & circular convolution

* Linear convolution of two sequences $x(n)$ of 'L' number of samples and $h(n)$ of M number of samples produce a result $y(n)$ which contains $N = L + M - 1$ samples
* For circular convolution if $x(n)$ contains 'L' number of samples and $h(n)$ has M number of

Pblm Given $x(n) = \begin{cases} 1/4 & 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$.

Compute DFT of the sequence.

Soln

N-point DFT of sequence $x(n)$ is defined as

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad K = 0, 1, \dots N-1$$

$$x(n) = \{1/4, 1/4, 1/4\}$$

$$X(K) = \frac{1}{4}\left[1 + e^{-j\omega} + e^{-2j\omega}\right]_{/\omega = 2\pi K/N}$$

$$= \frac{1}{4} e^{-j\omega}\left[1 + 2\cos\omega\right]_{/\omega = 2\pi K/N}$$

$$= \frac{1}{4} e^{-2\pi K/3}\left[1 + 2\cos\frac{2\pi K}{3}\right]$$

$$X(K) = \frac{1}{4} e^{-j2\pi nk/3}$$

$$= \left[1 + 2\cos\left(\frac{2\pi K}{3}\right)\right] \quad \text{where } K = 0, 1, \dots N-1$$

Ex:2 Derive the DFT of sample data sequence $x(n) = \{1, 1, 2, 2, 3, 3\}$ compute soln

N-point DFT of finite duration sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad k = 0, 1 \ldots N-1$$

For $k=0$ $\quad X(0) = \sum_{n=0}^{5} x(n) e^{-j(0)} = \sum_{n=0}^{5} x(n)$

$$= 1+1+2+2+3+3 = 12$$

for $k=1$, $X(1) = \sum_{n=0}^{5} x(n) e^{-j2\pi(1)n/6} = \sum_{n=0}^{5} x(n) e^{-jn\pi/3}$

$$= 1 + e^{-j\pi/3} + 2e^{-j2\pi/3} + 2e^{-j\pi} + 3e^{-j4\pi/3} + 3e^{-j5\pi/3}$$

$$= \cancel{1+[0.5} = -1.5 + \cancel{j0.866} + j2.598$$

for $k=2$

$$X(2) = \sum_{n=0}^{5} x(n) e^{-j2\pi(2)n/6\,3}$$

$$= \sum_{n=0}^{5} x(n) e^{-j2\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + 2e^{-j4\pi/3} + 2e^{-j2\pi} + 3e^{-j8\pi/3}$$

$$+ 3e^{-j10\pi/3}$$

$$= -1.5 + j0.866$$

For $k = 3$,

$$X(3) = \sum_{n=0}^{5} x(n) e^{-j2\pi(3)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-jn\pi}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$

$$= 0$$

For $k=4$

$$X(4) = \sum_{n=0}^{5} x(n) e^{-j2\pi(4)n/6} \quad \sum^{5} e^{-j4\pi n/6}$$

For $k=5$

$$X(5) = \sum_{n=0}^{5} x(n) e^{-j 2\pi(5)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-j 5\pi n/3}$$

$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{+j5\pi} + 3e^{j20\pi/3} + 3e^{j25\pi/3}$$

$$= -1.5 - j2.598$$

$$X(k) = \{12, -1.5+j2.598, -1.5+j0.866, 0, -1.5-j0.866, -1.5-j2.598\}$$

Amplitude Spectrum

$$|X(k)| = \{\sqrt{12^2}, \sqrt{(-1.5)^2+(-2.598)^2}, \sqrt{(-1.5)^2+(0.866)^2}, 0, \sqrt{(-1.5)^2+(-0.866)^2}, \sqrt{(-1.5)^2+(-2.598)^2}\}$$

$$= \{12, 2.99, 1.732, 0, 1.732, 2.99\}$$

Phase spectrum

$$\angle X(k) = \{\tan^{-1} 0, \tan^{-1}\left(\frac{2.598}{-1.5}\right), \tan^{-1}\left(\frac{0.866}{-1.5}\right), \tan^{-1}(0), \tan^{-1}\left(\frac{-0.866}{-1.5}\right), \tan^{-1}\left(\frac{-2.598}{-1.5}\right)\}$$

$$\angle X(k) = \{0, -\pi/3, -\pi/6, 0, \pi/6, \pi/3\}$$

Ex: Compute DFT of following finite length sequences considered to be of length 'N'

i) $x(n) = \delta(n)$ ii) $x(n) = \delta(n-n_0)$

Soln $X(\omega) = \sum^{N-1} x(n) e^{-j\omega n}$

Ex:3 Find N point DFT of $x(n) = a^n$ for $0 < a < 1$

Soln

N-point DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \qquad k = 0, 1 \ldots N-1$$

$$= \sum_{n=0}^{N-1} a^n e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} \left(a e^{-j2\pi k/N}\right)^n$$

$$= \frac{1 - \left(a e^{-j2\pi k/N}\right)^N}{1 - a e^{-j2\pi k/N}}$$

Formula

$$\boxed{\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}}$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j2\pi k/N}}$$

$$\because e^{-j2\pi k} = 1$$

$$= \frac{1 - a^N (1)}{1 - a e^{-j\frac{2\pi k}{N}}} \qquad k = 0, 1, 2 \ldots N-1$$

Ans $\boxed{X(k) = \frac{1 - a^N}{1 - a e^{-j\frac{2\pi k}{N}}}}$ $\quad k = 0, 1, 2 \ldots N-1$

Ex:4 Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$

(0.25 1 0.5 0.5)

Soln

IDFT is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \qquad n = 0, 1, 2 \ldots N-1$$

$$N = 4, \quad x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{\frac{j2\pi nk}{4}} \qquad n = 0, 1, 2, 3$$

When $n = 0$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{0}$$

$$= [X(0) + X(1) + X(2) + X(3)]$$

When $n=1$, $x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi 1 k/2}$

$$= \frac{1}{4}\left[1 + 2e^{j\pi/2} + 3e^{j\pi} + 4e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[-2 - j2\right] = -\tfrac{1}{2} - j\tfrac{1}{2}$$

When $n=2$,

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j\pi 2k/2}$$

$$= \frac{1}{4}\left[1 + 2e^{j\pi} + 3e^{j\pi(2)} + 4e^{j\pi(3)}\right]$$

$$= \frac{1}{4}\left[-2\right] =$$

$$= -\tfrac{1}{2}$$

When $n=3$,

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k)\, e^{j3\pi k/2}$$

$$= \frac{1}{4}\left[1 + 2e^{j3\pi/2} + 3e^{j3\pi} + 4e^{j9\pi/2}\right]$$

$$= \tfrac{1}{4}\left[-2 + 2j\right]$$

$$x(3) = -\tfrac{1}{2} + j\tfrac{1}{2}$$

Ans IDFT $x(n) = \left\{ \tfrac{5}{2},\ -\tfrac{1}{2} - j\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2} + j\tfrac{1}{2} \right\}$

H.W. Pblms

① Find the IDFT of $X(k) = \{3, 2+j, 1, 2-j\}$

② Find the 4-point DFT of sequence

$x(n) = \cos \frac{n\pi}{}$

PROPERTIES BASED PROBLEMS

CIRCULAR CONVOLUTION

3 methods : (1) Concentric circle Method
(2) Matrix Multiplication
(3) DFT IDFT Method.

Method 3 : DFT IDFT Method.

Consider two sequences $x_1(n)$, $x_2(n)$ which are of finite duration.

step 1) Let $X_1(k)$ and $X_2(k)$ be the N-point DFTs of the two sequences respectively

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} \qquad k=0,1,\ldots N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi nk}{N}} \qquad k=0,1,\ldots N-1$$

step 2 Let $x_3(n)$ be another sequence of length 'N' and its N point DFT $X_3(k) = X_1(k) X_2(k)$

step 3 $x_3(n)$ can be obtained by taking the IDFT of $X_3(k)$

ie $x_3(n) = IDFT\left[X_3(k)\right]$

Method 2 : Matrix Multiplication Method

$x(n) * h(n)$

$$\begin{bmatrix} h(0) \\ h(1) \\ \vdots \end{bmatrix} * \begin{bmatrix} x(0) & x(N-1) & x(N-2) & \cdots & x(2) & x(1) \\ x(1) & x(0) & x(N-1) & \cdots & x(3) & x(2) \\ x(2) & x(1) & x(0) & & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \end{bmatrix}$$

Ex:1 Compute the circular periodic convolution of the two sequences

$x_1(n) = \{1, 1, 2, 2\}$

$x_2(n) = \{1, 2, 3, 4\}$

soln

Method 1 : Diagram method

Circular periodic convolution

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n, (\text{mod } N))$$

$m = 0, 1 \ldots N-1$

m=0

$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2(-n)$$

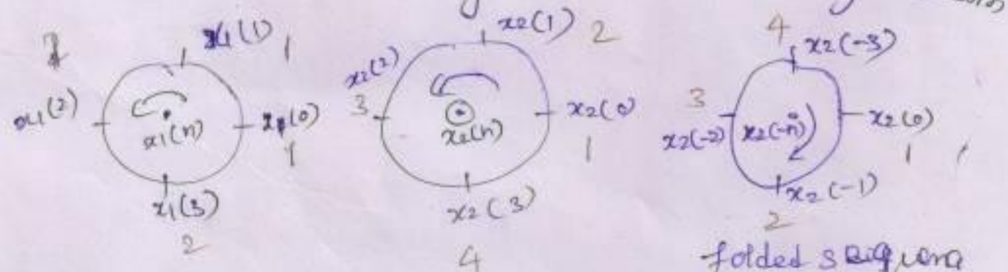
$x_2(0) = x_2(0)$

$x_2(-1) = x_2(3)$

$x_2(-2) = x_2(2)$

$x_2(-3) = x_2(1)$

$x_3(0)$ obtained by computing product sequence of $x_1(n)$ & $x_2(-n)$ point by point & taking the sum

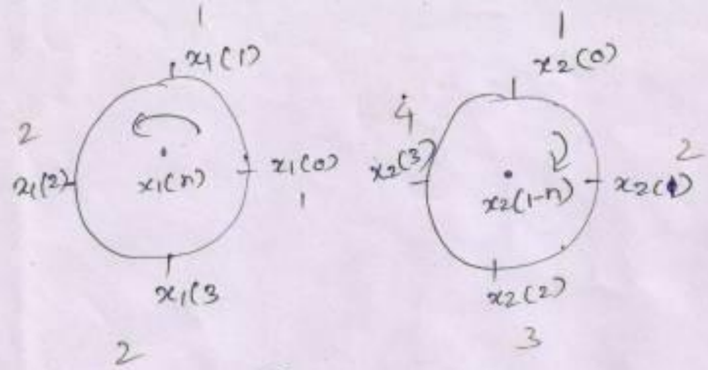


$$x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2(-n) = 1\times1 + 1\times4 + 2\times3 + 2\times2$$

$= 15$

Similarly for $m = 1$

$$x_3(1) = \sum_{n=0}^{N-1} x_1(n)\, x_2(1-n)$$



$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2(1-n)$$

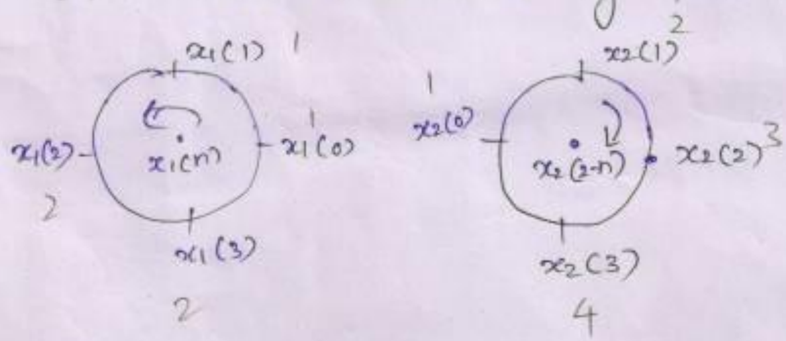
$$= 2\times1 + 1\times1 + 4\times2 + 3\times2$$

$$= 17$$

When $m=2$

$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2(2-n)$$

$x_2(2-n)$ can be obtained by rotating $x_2(n)$ counter clockwise by two units in time

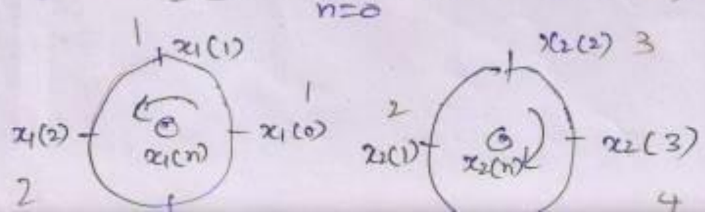


$$x_3(2) = 1\times3 + 1\times2 + 2\times1 + 2\times4$$

$$= 15$$

When $m=3$

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2(3-n)$$



Method 2

Matrix Multiplication method.

Pblm

compute $x_1(n) \, (N) \, x_2(n)$ if

$x_1(n) = \delta(n) + \delta(n-1) - \delta(n-2) - \delta(n-3)$

$x_2(n) = \delta(n) - \delta(n-2) + \delta(n-4)$

Given $N=5$

Soln

$x_1(n) = \{1, 1, -1, -1\}$

$x_2(n) = \{1, 0, 1, 0, 1\}$.

Gn

$N=5$

for finding circular convln two i/p sequences length must be same.

~~So $N_1 + N_2 - 1 = 4+5-1 = 8$.~~

So add one zero to the sequence $x_1(n)$ to bring its length to 5

$x_1(n)$ becomes $= \{1, 1, -1, -1, 0\}$.

Formula for circular Convolution is

$$\begin{bmatrix} x_1(0) & x_1(N-1) & x_1(N-2) & \cdots & x_1(1) \\ x_1(1) & x_1(0) & x_1(N-1) & \cdots & x_1(2) \\ \vdots & & & & \\ x_1(N-1) & x_1(N-2) & x_1(N-3) & \cdots & x_1(0) \end{bmatrix} \begin{bmatrix} x_2(0) \\ x_2(1) \\ \vdots \\ x_2(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ \vdots \\ x_3(N-1) \end{bmatrix}$$

$$x_3(n) = \begin{bmatrix} 3 \\ 0 \\ -3 \\ -2 \\ 2 \end{bmatrix}$$

Ans $x_3(n) = x_1(n) \textcircled{N} x_2(n)$

$= \{3, 0, -3, -2, 2\}$
$\uparrow$

Method 3 : Circular convolution using DFT, IDFT Method

Pblm Compute circular convolution of the two sequences using DFT, IDFT

Soln

$X_3(k) = X_1(k) \; X_2(k)$

$x_3(n) = IDFT\,[X_3(k)]$

Step 1 : To find $X_1(k)$

Given $x_1(n) = \{1, 1, 2, 2\}$

N-point DFT $X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}$ $\quad k = 0, 1 \dots N-1$

$N = 4$ $\quad X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi nk/4}$ $\quad k = 0, 1 \dots 3$

$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j\pi nk/2}$

When $k=1$, $X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n/2}$

$= -1+j$

When $k=2$, $X_1(2) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n}$

$= 0$

When $k=3$; $X_1(3) = \sum_{n=0}^{3} x_1(n) e^{-j\frac{3n\pi}{2}}$

$= -1-j$

$X_1(k) = \{6, -1+j, 0, -1-j\}$

Step 2: To find $X_2(k)$

$x_2(n) = \{1, 2, 3, 4\}$

Give N-Point DFT $X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{2\pi nk}{4}}$

$k = 0, 1, 2, 3$

When $k=0$, $X_2(0) = \sum_{n=0}^{3} x_2(n) e^{0}$

$= 10$

When $k=1$, $X_2(1) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{\pi n 1}{2}}$

$= -2+2j$

When $k=2$, $X_2(2) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{\pi n 2}{2}}$

$= -2$

When $k=3$, $X_2(3) = \sum_{n=0}^{3} x_2(n) e^{-j\frac{\pi n 3}{2}}$

$X_2(k) = \{10, -2+2j, -2, -2-2j\}$

Step 3: To find $X_3(k)$

$X_3(k) = X_1(k) \; X_2(k)$

$= \{6, -1+j, 0, -1-j\} \; \{10, -2+2j, -2, -2-2j\}$

$= \{60, -4j, 0, 4j\}$

Step 4: To find $x_3(n)$

$x_3(n) = \text{IDFT}[X_3(k)]$

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi nk/N} \quad n = 0, 1 \ldots N-1$$

$N = 4$

$$x_3(n) = \frac{1}{4}\sum_{k=0}^{3} X_3(k)\, e^{\frac{j\cancel{2}\pi nk}{\cancel{4}2}}$$

When $n=0$, $x_3(0) = \frac{1}{4}\left[\cancel{60}\; X(k)e^{0}\right]$

$= \frac{1}{4}[60] = 15$

When $n=1$, $x_3(1) = \frac{1}{4}\sum_{k=0}^{3} X_3(k)e^{\frac{j\pi 1k}{2}}$

$= \frac{1}{4}[68] = 17$

When $n=2$, $x_3(2) = \frac{1}{4}\sum_{k=0}^{3} X_3(k)e^{\frac{j\pi \cancel{2}k}{\cancel{2}}}$

$= \frac{1}{4}[60] = 15$

When $n=3$, $x_3(3) = \frac{1}{4}\sum_{k=0}^{3} X_3(k)e^{\frac{j\pi 3k}{2}}$

To find linear convolution

Method 1 : formula

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Method 2 : Matrix Representation

$$\begin{array}{c|ccc} x(n) \backslash h(n) & h(0) & h(1) & \cdots \\ \hline x(0) & P(0) & P(1) & P(2) \\ x(1) & P(3) & P(4) & \cdots \\ x(2) & P(5) & & \end{array}$$

$$y(n) = \{P(0),\ P(3)+P(1),\ P(5)+P(4)+P(2) \ldots\}$$

To find linear convolution through circular convolution with padding of zeros

Pblm

Find linear conv^n thro circular conv^n for the sequence given below

$$h(n) = \{1, 2, 4\}$$
$$x(n) = \{1, 2\}$$

$N_1 = 3$, $N_2 = 2$, $N_1 + N_2 - 1 = 4$

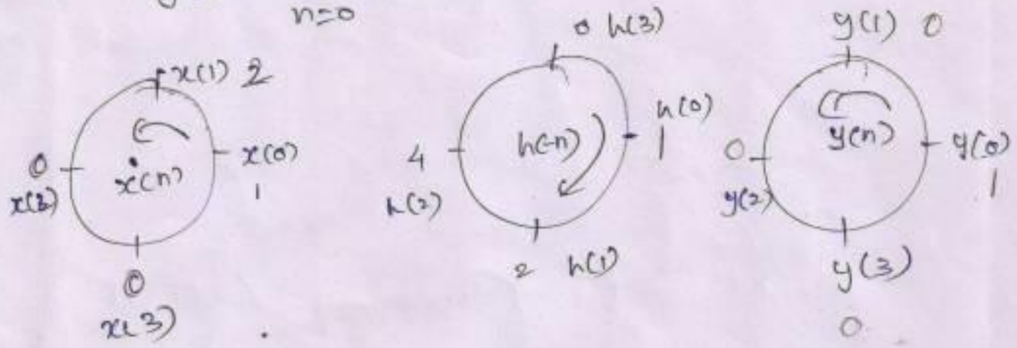
Append $N_2 - 1$ zeros to $h(n)$

" $N_1 - 1$ zeros to $x(n)$

$$h(n) = \{1, 2, 4, 0\}$$

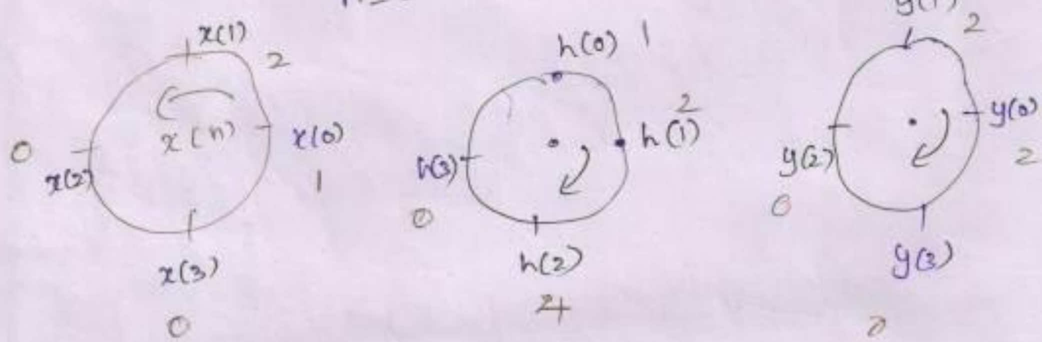
$n=0,$

$$y(0) = \sum_{n=0}^{3} x(n)\, h(-n)$$



$$y(0) = 1+0+0+0 = 1$$

$$n=1, \quad y(1) = \sum_{n=0}^{3} x(n)\, h(1-n)$$



$$y(1) = \{2+2+0+0\} = 4$$

Similarly

for $n=2$, $\quad y(2) = \sum_{n=0}^{3} x(n)\, h(2-n)$

$$= \{4+4+0+0\} = 8$$

For $n=3$, $\quad y(3) = \sum_{n=0}^{3} x(n)\, h(3-n)$

$$= \{0+8+0+0\} = 8$$

Ans $\boxed{y(n) = \{1, 4, 8, 8\}}$

~~FAST FOURIER TRANSFORM~~

~~FFT computations using~~

RELATIONSHIP BETWEEN DTFT & DFT

DTFT of a sequence can be written as $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

If the sequence is of finite duration of length 'N'

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \longleftarrow (1)$$

DFT of finite duration sequence $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \longleftarrow (2) \qquad k=0,1,\ldots N-1$$

Compare eqn (1) (2)

$$\boxed{X(k) = X(e^{j\omega})}\Big|_{\omega=\frac{2\pi k}{N}} \longleftarrow (3) \qquad k=0,1\ldots N-1$$

eqn (3) gives relnship b/w DTFT & DFT

Relationship b/w DFT & Z transform

Z transform of finite duration sequence $x(n)$ can be written as

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n} \longleftarrow (4)$$

Compare eqn (1) (2)

FAST FOURIER COMPUTATION USING DIT & DIF ALGORITHM

FFT - algorithm that efficiently computes DFT

DFT of sequence $\{x(n)\}$ of length 'N' is given by a complex valued sequence $\{X(k)\}$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \qquad 0 \le k \le N-1$$

W_N be the complex valued phase factor which is an N-th root of unity expressed by

$$W_N = e^{\frac{-j2\pi}{N}}$$

Sub W_N in $X(k)$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad 0 \le k \le N-1}$$

FFT

IDFT becomes

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad 0 \le k \le N-1$$

For each value of k, direct computation of $X(k)$ involves

'N' complex multiplications

N-1 Complex Additions.

To compute all 'N' values of DFT

N^2 complex multiplications

$N(N-1)$ complex additions are required

An efficient algorithm for DFT computation is the FFT algorithm

FFT algorithm exploits symmetry & periodicity property

Symmetry property $W_N^{k+N/2} = -W_N^k$

Periodicity property $W_N^{k+N} = -W_N^k$

Radix

- Divide and conquer approach used to develop DFT algorithm

- decomposition of N-point DFT into successively smaller size DFTs.

If N is factored as $N = r_1 r_2 r_3 \ldots r_L$

$r_1 = r_2 = \ldots = r_L = r$ the $\boxed{N = r^L}$

So the DFT will be of size 'r'. The number 'r' is called radix of FFT algorithm.

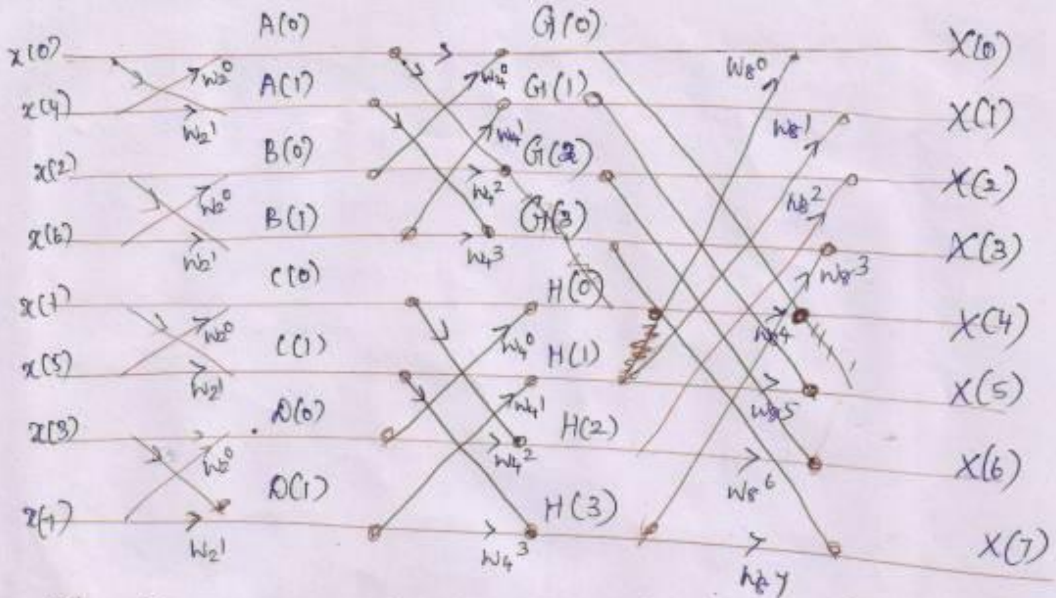
Most widely used is radix-2 FFT algorithm

FFT Algorithm Types

1. DIT algorithm
2. DIF algorithm.

DIT algorithm :

DIT FFT algorithm for N=8 flowgraph



(fig) flow graph of DITFFT algorithm for N=8

DITFFT algorithm for N=8 flow graph consists of 3 stages

I stage : computes four 2-point DFTs

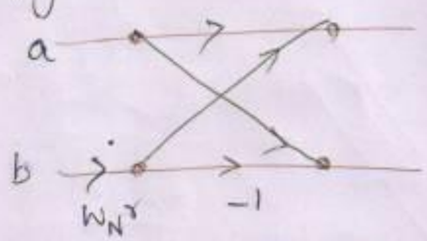
II stage : computes two 4-point DFTs

III stage : computes 8-point DFT

I/p data has been shuffled (bit reversed order

Index	Binary representation	Bit reversed Binary	Bit reversed Index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1

Basic computation in DIT FFT algorithm is called butterfly diagram because the shape of the flow graph resembles a butterfly



$A = a + W_N^r b$

$B = a - W_N^r b$

(fig) Basic butterfly flow graph for DIT FFT

Total number of complex multiplications and additions in computing all N-DFT samples $= N \log_2 N$

$8 \log_2 8$
$= 8 \times 3$
$= 24$

complex multiplications and addition for computing all N-DFT samples using direct computation is $N^2, N(N-1)$

So complex multiplications reduced from N^2 to $N \log_2 N$

Complex additions reduced from N to $N/\log_2 N$

Problems using DIT FFT algorithm

Pblm 1: Given $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ Find $X(k)$ using DIT FFT algorithm.

Soln Given $N = 8$, ... phase factor W_N^k

$N = 8.$

$$W_8^0 = e^{-j(2\pi/8)0} = e^0 = 1$$

$$W_8^1 = e^{-j(2\pi/8)1} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4$$

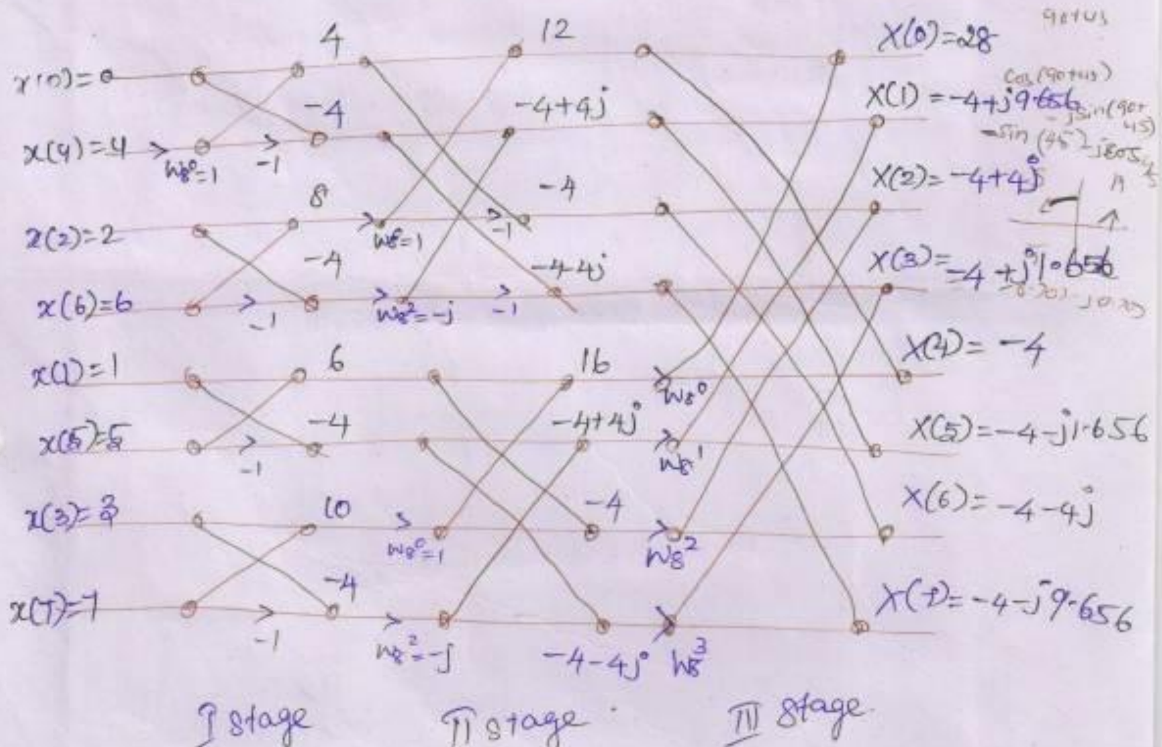
$$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j(2\pi/8)2} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2$$

$$= -j$$

$$W_8^3 = e^{-j\,2\pi/8(3)} = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4$$

$$= -0.707 - j0.707$$



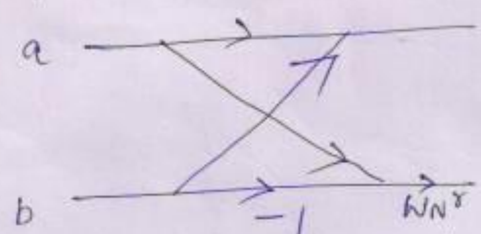
$$X(k) = \{28, -4+j9.656, -4+4j, -4-j1.656,$$
$$-4, -4-j1.656, -4-4j, -4-j9.656\}$$

DIF - Decimation in frequency Algorithm

DIFFT algorithm decomposes DFT by sequentially splitting i/p samples $x(n)$ in the time domain into sets of smaller and smaller subsequences and then forms a weighted combination of the DFTs of these subsequences.

DIFFT decomposes DFT by recursively splitting the sequence elements $X(k)$ in the frequency domain into sets of smaller and smaller subsequences.

For computing N-point DFT down to 2-point DFT DIF FFT algorithm needs $(N/2)\log_2 N$ multiplications & need ~~complex~~ m $N \log_2 N$ complex additions



(fig) Basic butterfly for DIFFT

For DIF FFT algorithm

Soln

$$W_N^k = e^{-j(2\pi/N)k}$$

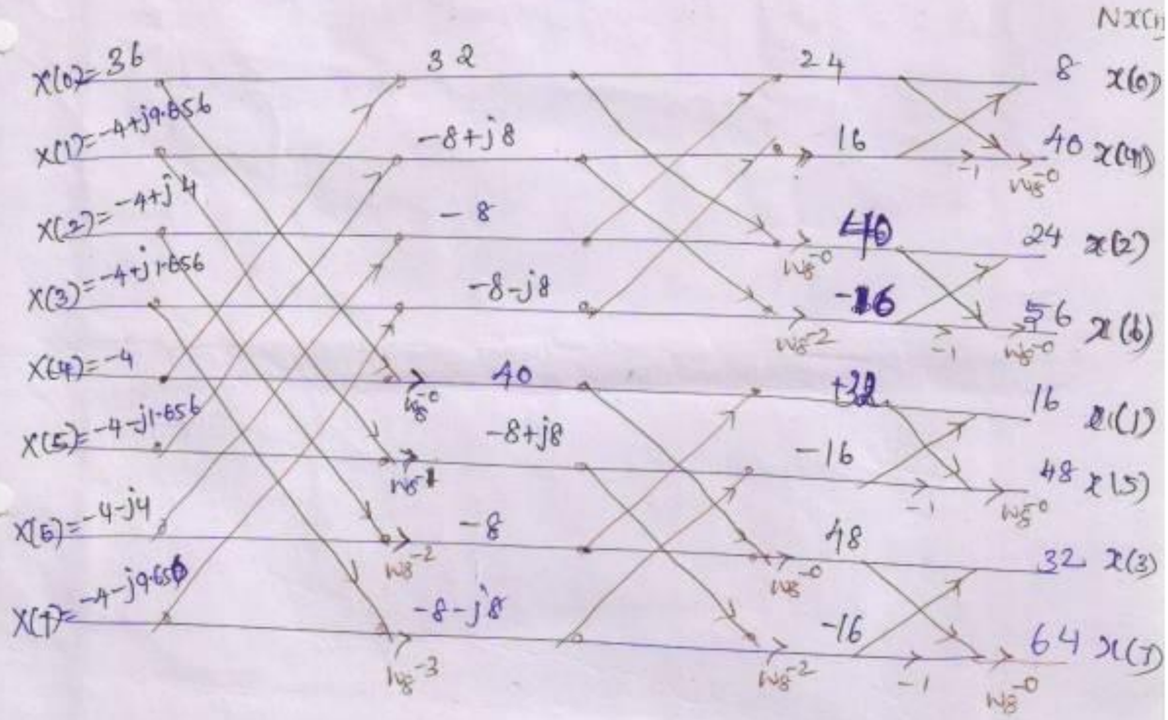
$N=8$

$$W_8^{-0} = e^0 = 1$$

$$W_8^{-1} = e^{-j(2\pi/8)(-1)} = e^{j\pi/4} = 0.707+j0.707$$

$$W_8^{-2} = e^{-j(2\pi/8)-2} = e^{j\pi/2} = +j$$

$$W_8^{-3} = e^{-j(2\pi/8)(-3)} = e^{j3\pi/2} = -0.707+j0.707$$



$$x(n) = \frac{1}{N}\{8, 16, 24, 32, 40, 48, 56, 64\}$$

$N \to 8$

Ans. $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

IDFT - ~~Problems~~ Inverse Discrete Fourier Transform

Inverse DFT is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} \qquad n=0,1 \cdots N-1 \qquad (1)$$

Take complex conjugate of eqn (1)

$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}$$

R.H.S is the DFT of the sequence $X^*(k)$

$$\therefore\ x^*(n) = \frac{1}{N}\, DFT\left[X^*(k)\right]$$

Take complex conjugate of both sides, we will get the desired o/p $x(n)$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}\right]^*$$

$$\boxed{x(n) = \frac{1}{N}\left[FFT\left[X^*(k)\right]\right)^*} \rightarrow \text{IDFT formula}$$

FFT algorithm can be used to calculate IDFT if output is multiplied by N & twiddle factors are -ive powers of W_N.

pblm 1

$$X(k) = \{36, -4+j9.656, -4+j4, -4+j1.656, -4, -4-j1.656, -4-j4, -4-j9.656\}$$

Pblm

Given $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

Find $X(k)$ using DIF FFT algorithm

Soln

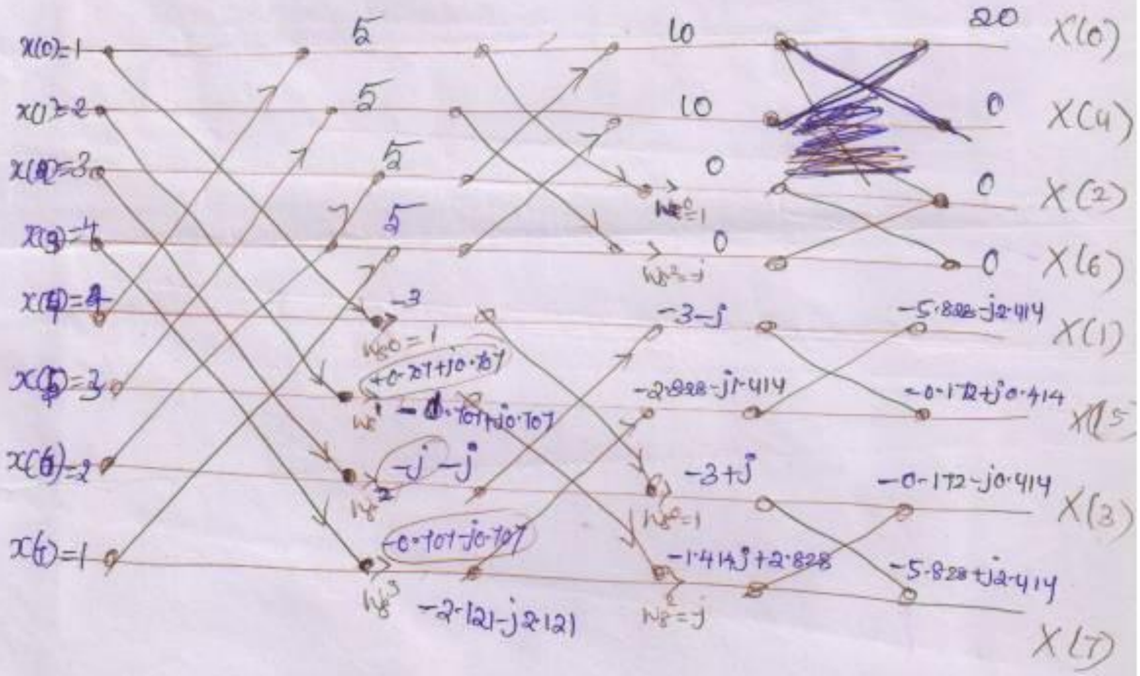
$N = 8, \quad W_N^k = e^{-j(2\pi/N)k}$ $\quad e^{-j\omega t}$

$W_8^0 = e^0 = 1$

$W_8^1 = e^{-j(2\pi/8)1} = e^{-j\pi/4} = 0.707 - j0.707$

$W_8^2 = e^{-j(2\pi/8)2} = e^{-j\pi/2} = -j$

$W_8^3 = e^{-j(2\pi/8)3} = e^{-j3\pi/4} = -0.707 - j0.707$



Ans $X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$.

OVER LAP ADD & SAVE METHOD

Fast convolution (Sectioned convolution)

During implementing linear convolution in FIR filter

* i/p sequence x(n) is longer than h(n) of DSP System.
* circular convolution used to implement linear convolution by padding 0's

Why fast convolution,

1) output can't be obtained until entire i/p signal is received & so there will be characteristic delay.

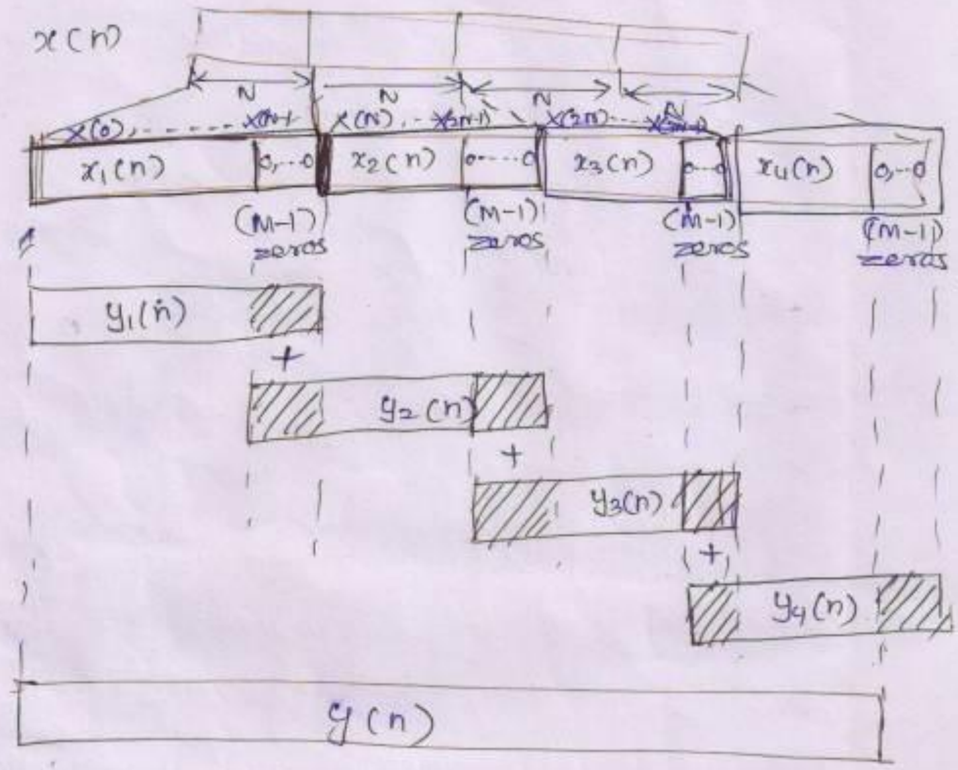
2) As $N_1 + N_2 - 1$ signal gets longer FFT implementation & size of memory becomes impractical

To eliminate all these problems while performing filtering (convolution) operations in frequency domain ~~and~~ two segmentation methods are used to perform fast convolution

1) Overlap - add method

2) Overlap - Save Method.

Overlap Add Method



Steps

1) (N-1) zeros padded at the end of h(n) of length M & sequence of length M+N-1 = L is obtained.

2) An L-point FFT on ~~each~~ the selected data block is performed. Each data block has 'N' i/p data values & (M-1) zeros

3) FFT of sequence obtained in step ① is multiplied by FFT of sequence of the selected data block obtained in step ②

5) The first (M-1) IFFT values obtained in step ④ is overlapped with last (M-1) IFFT values for previous block. Add to produce the final convolution o/p sequence y(n)

6) For the next data block, go to step ③

Pblm 1:

An FIR filter has the unit impulse response sequence $h(n) = \{2, 2, 1\}$. Determine o/p sequence in response to the i/p sequence $x(n) = \{3, 0, -2, 0, 2, 1, 0, -2, -1, 0\}$ using overlap-add convolution method.

Soln

Given $h(n) = \{2, 2, 1\}$. $M = 3$.

Length of FFT operation is selected as

$L = 2^M = 2^3 = 8$

$\therefore N = L - M + 1$
$= 8 - 3 + 1$
$= 6$.

(M-1

length of data block = 6.

~~By using another mek~~

n	0	1	2	3	4	5	6	7,	8	9	10	11	12	13

Another method

$$x(n) = \{3, 0, -2 \,|\, 0, 2, 1 \,|\, 0, -2, -1 \,|\, 0\}$$

$$h(n) = \{2, 2, 1\}$$

let the length of each block in $x(n)$ be

$L = 3$

length of $h(n) = 3$

Total length of data block $= L + M - 1$
$= 3 + 3 - 1$
$= 5$

Padd two zeros to the end of each block in $x(n)$ to bring the data block length to 5

$$x_1(n) = \{3, 0, -2, \underbrace{0, 0}_{(M-1)\text{ zeros padded}}\}$$

$$x_2(n) = \{0, 2, 1, 0, 0\}$$

$$x_3(n) = \{0, -2, -1, 0, 0\}$$

$$x_4(n) = \{0, 0, 0, 0, 0\}$$

$$h(n) = \{2, 2, 1, 0, 0\}$$

Now perform circular convolution b/w i/p data blocks and $h(n)$

$$x_1(n) \text{ⓝ} h(n) = \{3, 0, -2, 0, 0\} \text{ⓝ} \{2, 2, 1, 0, 0\}$$

$$y_1(n) = \{6, 6, -1, -4, -2\}$$

$$x_2(n) \text{ⓝ} h(n) = \{0, 2, 1, 0, 0\} \text{ⓝ} \{2, 2, 1, 0, 0\}$$

$$y_2(n) = \{0, 4, 6, 4, 1\}$$

M xions, M/2 → linear, M-1 addition

Comparision of the efficiency of FFT alg c̄. direct form realization of FIR filter

If $M = 128 = 2^7$, $N = 2^v$.

Then no of complex xions per o/p pt for FFT size of $N = 2^v$ is

$$C(v) = \frac{N \log_2 N}{N - M + 1} = \frac{2^v (v+1)}{2^v - 2^7}$$

$$\approx \frac{2^v (v+1)}{2^v - 2^7}$$

$C(v) \leftarrow$ no of complex xions for FFT based method

– no of real xions = 4 times this number

– η of FFT method can be improved by computing DFT of 2 successive data blocks simultaneous

– FFT based method is superior from comp. pt of view, when filter length is relatively large

Computational Complexity

Size of FFT $v = \log_2 N$	C(v) No of Complex xions per o/p pt
9	13.3
10	12.6
11	12.8
12	13.4
14	15.1

ns

No of stages M	No of points N	No of complex xions using Direct evaluation N^2	No of complex xions using FFT alg $\frac{N}{2}\log_2 N$	Speed Improvement factor $\frac{N^2}{\frac{N}{2}\log_2 N}$
2	4	16	4	4
3	8	64	12	5.333
4	16	256	32	8
5	32	1024	80	12.8
6	64	4096	192	21.33
7	128	16384	448	36.57
8	256	65536	1024	64
9	512	262144	2304	113.77
10	1024	1048576	5120	204.8